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Approximation by Zygmund means in variable exponent Lebesque spaces

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Abstract. In the present work we investigate the approximation of the functions by the Zygmund means in variable exponent Lebesgue spaces. Here the estimate which is obtained depends on sequence of the best approximation in Lebesgue spaces with variable exponent. Also, these results were applied to estimates of approximations of Zygmund sums in Smirnov classes with variable exponent defined on simply connected domains of the complex plane.

1. Introduction and the Main Results

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$. Let us denote by $\wp$ the class of Lebesgue measurable functions $p:\mathbb{T}\longrightarrow(1,\infty)$ such that $1<p_*:=\underset{x\in\mathbb{T}}{ess\ \inf}p(x)\leq p^*:=\underset{x\in\mathbb{T}}{ess\ \sup}\, p(x)<\infty$. The conjugate exponent of $p(x)$ is shown by $p'(x):=\frac{p(x)}{p(x)-1}$. For $p\in\wp$, we define a class $L^{p(\cdot)}(\mathbb{T})$ of 2π periodic measurable functions $f:\mathbb{T}\to\mathbb{R}$ satisfying the condition

$$\int_{\mathbb{T}}|f(x)|^{p(x)}\,dx<\infty.$$

This class $L^{p(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{T})}:=\inf\{\ \lambda>0:\int_{\mathbb{T}}\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx\leq 1\}.$$

We say that the variable exponent $p(x)$ satisfies *local log- continuity condition,* if there is a positive constant c_1 such that

$$|\ p(x)-p(y)\ |\leq\frac{c_1}{-\ln|x-y|}, \tag{1}$$

for all $x,y\in\mathbb{T}$, $|x-y|<\frac{1}{2}$.

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A function $p \in \wp$ is said to belong to the class $\wp^{\log}$, if the condition (1) is satisfied.The spaces $L^{p(\cdot)}(\mathbb{T})$ are called generalized Lebesgue spaces with variable exponent. It is know that for $p(x) := p$ $(0 < p \leq \infty)$, the space $L^{p(x)}(\mathbb{T})$ coincides with the Lebesgue space $L^{p}(\mathbb{T})$. For the first time Lebesgue spaces with variable exponent were introduced by Orlicz [26]. Note that the generalized Lebesgue spaces with variable exponent are used in the theory of elasticity, in mechanics, especially in fluid dynamics for the modelling of electrorheological fluids, in the theory of diferential operators, and in variational calculus [4], [6], [7] and [28]. Detailed information about properties of the Lebesque spaces with variable exponent can be found in [8], [24] and [31]. Note that, some of the fundamental problems of the approximation theory in the generalized Lebesgue spaces with variable exponent of periodic and non-periodic functions were studied and solved by Sharapudinov [32]-[35].

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x,f), \ A_k(x,f) := a_k(f)\cos kx + b_k(f)\sin kx \tag{2}$$

be the Fourier series of the function $f \in L_1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f . The $n-th$ *partial sums*, *Zygmund means of order* k $(k \in \mathbb{N})$ of the series (2) are defined, respectively as [12], [36]:

$$\begin{aligned} S_n(x,f) &= \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x,f), \\ Z_{n,k}(x,f) &= \frac{a_0}{2} + \sum_{\nu=1}^{n} \left(1 - \frac{\nu^k}{(n+1)^k}\right) A_\nu(x,f), \ k = 1,2,..., \ n = 1,2,... \end{aligned}$$

It is clear that

$$S_0(x,f) = Z_{0,k}(x,f) = \frac{a_0}{2}.$$

For $f \in L^{p(\cdot)}(\mathbb{T})$ we define the Steklov operator by

$$s_h(f)(x) = \frac{1}{h}\int_{x}^{x+h} f(t)dt = \frac{1}{h}\int_{0}^{h} f(u+x)du$$

and the $k-th-$modulus of smoothness $\Omega_k(f,\cdot,)_{p(.)}$ $(k = 1,2,...)$ by

$$\Omega_k(f,\delta)_{p(.)} = \sup_{\substack{0<h_i\leq\delta \\ 1\leq i\leq k}} \left\| \prod_{i=1}^{k} (I - s_{h_i})(f) \right\|_{L^{p(\cdot)}(\mathbb{T})} \qquad \delta > 0,$$

where I is the identity operator. Note that the $k-th-$ modulus of continuity $\Omega_k(f,\cdot)_{p(.)}$ is a nondecreasing, nonnegative, continuous function and

$$\lim_{\delta\to 0}\Omega_k(f,\cdot)_{p(.)}=0,\quad \Omega_k(f+g,\cdot)_{p(.)}\leq \Omega_k(f,\cdot)_{p(.)}+\Omega_k(g,\cdot)_{p(.)},$$

for $f,g\in L^{p(\cdot)}(\mathbb{T})$.

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let $G^{-}:=ext\Gamma$. We denote

$$\mathbb{T}^*:=\{w\in\mathbb{C}:|w|=1\},\ \mathbb{D}:=int\,\mathbb{T}^*\ ,\ \mathbb{D}^-:=ext\,\mathbb{T}^*.$$

Let $w=\varphi(z)$ be the conformal mapping of G^- onto $\mathbb{D}^-$ normalized by

$$\varphi(\infty)=\infty,\quad \lim_{z\to\infty}\frac{\varphi(z)}{z}>0$$

and let ψ denote the inverse of φ.

For any measurable bounded exponent $p(z)\geq 1$ we denote by $L^{p(\cdot)}(\Gamma)$ the set of functions f, such that

$$\int_\Gamma |f(z)|^{p(z)}\,|dz|<\infty$$

and

$$\|f\|_{L^{p(\cdot)}(\Gamma)}:=\inf\left\{\alpha>0:\int_\Gamma\left|\frac{f(z)}{\alpha}\right|^{p(z)}|dz|\leq 1\right\}.$$

We denote by K segment $[0,2\pi]$ or Jordan rectifiable curve in the complex plane $\mathbb{C}$. We suppose that Lebesgue measurable function $p(.):K\to[0,\infty)$ satisfies the following conditions:

$$1\leq p_*:=\underset{z\in K}{ess}\inf p(z)\leq p^*:=\underset{z\in K}{ess}\sup p(z)<\infty. \tag{3}$$

If $p(.)$ satisfies the conditions (3) and

$$|p(z_1)-p(z_2)|\leq\frac{c}{\ln\left(\frac{|K|}{|z_1-z_2|}\right)},$$

we say that $p(.)\in\Phi^{\log}(K)$,where $|K|$ is the Lebesgue measure of K. A function p belong to the class $\Phi_0^{\log}(K)$ if $p(.)\in\Phi^{\log}(K)$ with $p_*>1$ [18].

We define also the variable exponent *Smirnov class* $E_{p(.)}(G)$ as

$$E_{p(.)}(G):=\left\{f\in E_1(G):f\in L^{p(\cdot)}(\Gamma)\right\}.$$

For $f\in L^{p(\cdot)}(\Gamma)$ with $p\in\wp^{\log}$ we define the function

$$f_0(t):=f(\psi(t)),t\in\mathbb{T}^*,$$
$$p_0(t)=p(\psi(t)).$$

Let h be a continuous function on $[0, 2\pi]$. Its modulus of continuity is defined by

$$\omega(t,h) := \sup\{|h(t_1) - h(t_2)| : t_1, t_2 \in [0, 2\pi], |t_1 - t_2| \leq t\}, t \geq 0.$$

The curve Γ is called *Dini-smooth* if it has a parameterization

$$\Gamma : \varphi_0(s),\ 0 \leq s \leq 2\pi,$$

such that $\varphi_0'(s)$ is Dini-continuous, i.e.

$$\int_0^{\pi} \frac{\omega(t, \varphi_0')}{t} dt < \infty$$

and $\varphi_0'(s) \neq 0$ [27, p. 48].

If Γ is a Dini-smooth curve, then there exist [38] the constants $c_2,\ c_3,\ c_4$ and c_5 such that

(4)
$$0 \leq c_2 \leq \left|\psi'(t)\right| \leq c_3 < \infty,\ |t| > 1.\ 0 \leq c_4 \leq \left|\varphi'(z)\right| \leq c_5 < \infty,\ |t| > 1.$$

a.e. on $\mathbb{T}^*$ and on Γ, respectively. Note that if Γ is a Dini-smooth curve, then by (4) we have $f_0 \in L^{p_0(\cdot)}(\mathbb{T}^*)$ for $f \in L^{p(\cdot)}(\Gamma)$. It is known that [15], if Γ is a Dini-smooth curve, then $p_0 \in \Phi^{\log}(\mathbb{T})$ if and only if $p \in \Phi^{\log}(\Gamma)$.

Let Γ be a rectifiable Jordan curve and $f \in L^1(\Gamma)$. Then the functions f^+ and f^- defined by Γ,

$$f^+(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\psi'(w)}{\psi(w) - z} f_0(w)dw,\ z \in G \tag{5}$$

and

$$f^-(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\psi'(w)}{\psi(w) - z} f_0(w)dw,\ z \in G^-,$$

are analytic in G and G^- respectively, and $f^-(\infty) = 0$. Thus the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma \cap \{\zeta:\ |\zeta - z| > \varepsilon\}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

exists and is finite for almost all $z \in \Gamma$.

The quantity $S_\Gamma(f)(z)$ is called the *Cauchy singular integral* of f at $z \in \Gamma$.

According to the Privalov's theorem [9, p. 431] if one of the functions f^+ or f^- has the non-tangential limits a.e. on Γ, then $S_\Gamma(f)(z)$ exists a.e. on Γ and also the other one has the non-tangential limits a.e. on Γ. Conversely, if $S_\Gamma(f)(z)$ exists a. e. on Γ, then the functions $f^+(z)$ and $f^-(z)$ have non-tangential limits a.e. on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2} f(z),\ f^-(z) = S_\Gamma(f)(z) - \frac{1}{2} f(z)$$

and hence

$$f = f^{+} - f^{-}$$

holds a.e. on Γ.

Let $\varphi_k(z),\ k = 0,1,2,...$ be the Faber polynomials for G. The Faber polynomials $\varphi_k(z)$, associated with $G\cup\Gamma$, are defined through the expansion

$$\frac{\psi'(t)}{\psi(t)-z} = \sum_{k=0}^{\infty} \frac{\varphi_k(z)}{t^{k+1}},\ \ z\in G,\ t\in\mathbb{D} \tag{6}$$

and the equalities

$$\varphi_k(z) = \frac{1}{2\pi i}\int\limits_{|t|=R} \frac{t^k\psi'(t)}{\psi(t)-z}dt \ \ z\in G_R,\ R>1,$$

$$\varphi_k(z) = \varphi^k(z) + \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\varphi^k(s)}{s-z}ds, \quad z\in G^{-},$$

hold [30, p. 33-48].

Let $f\in E_{p(.)}(G)$. Since $f\in E_1(G)$,we have

$$f(z) = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f(s)}{s-z}ds = \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f(\psi(t))\psi'(t)}{\psi(t)-z}dt,$$

for every $z\in G$. Considering this formula and expansion (6), we can associate with f the formal series

$$f(z) \sim \sum_{k=0}^{\infty} c_k(f)\varphi_k(z), \tag{7}$$

where

$$c_k(f) := \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f(\psi(t))}{t^{k+1}}dt.$$

The series (7) is called the *Faber series* expansion of f, and the coefficients $c_k(f)$ are said to be the *Faber coefficients* of f.

The *Zygmund sums* of the series (6) is defined as

$$Z_{n,k}(z,f) = \sum_{\nu=0}^{n}(1-\frac{\nu^k}{(n+1)^k})c_k(f)\varphi_k(z).$$

Let $P :=\{$all polynomials (with no restriction on the degree)$\}$, and let $P(\mathbb{D})$ be the set of traces of members of P on $\mathbb{D}$. We define the operator

$$T:\ P(\mathbb{D})\to E_{p(.)}(G)$$

as

$$T(P)(z) := \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{P(w)\psi'(w)}{\psi(w)-z}dw, \quad z\in G.$$

Then using (6) we have

$$T\left(\sum_{k=0}^{\infty}\alpha_k w^k\right)=\sum_{k=0}^{\infty}\alpha_k\varphi_k(z),$$

where $\varphi_k(z),\ k\in\mathbb{N}$, are the Faber polynomials of $\overline{G}$. Use of (5) and (6) gives us Faber series representation

$$f^+(z)=\sum_{k=0}^{\infty}c_k(f)\varphi_k(z),$$

where

$$c_k(f)=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0(w)}{w^{k+1}},\quad k\in\mathbb{N}.$$

We shall use the $c, c_1, c_2, ...$ to denote constants (in general, different in different relations) depending only on quantities that are not important for the questions of interest.

We denote by $E_n(f)_{p(.)}$ the best approximation of $f\in L^{p(.)}(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_{p(.)}=\inf\{\| f-T_n\|_{L^{p(.)}(\mathbb{T})}: T_n\in\Pi_n\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

Note that the properties of Lebesgue spaces with variable exponents have been investigated intensively by many authors (see, for example, [4]-[8], [24] and [31]).

The approximation problems in non-weighted and weighted Lebesgue spaces with variable exponents were studied in [1], [2], [11], [15]-[19], [22], [32]-[35] and [37].

In this study we investigate the approximation of the functions by Zygmund means in variable exponent Lebesgue spaces. Note that estimates in this study are obtained in terms of the best approximation $E_n(f)_{p(.)}$ and modulus of smoothness. These results were applied to estimates of approximations of Zygmund sums in Smirnov classes with variable exponent defined on simply connected domains of the complex plane. Similar problems of the approximation theory in the different spaces have been studied by several authors (see, for example, [3], [10], [12]-[14], [20]-[23], [25], [29], [36] and [39]).

Note that for the proof of the new results obtained in the variable exponent Lebesgue spaces we apply the method developed in [10], [13] and [15].

Our main results are the following.

Theorem 1.1. *Let* $f \in L^{p(.)}(\mathbb{T}),\ r \in Z_+,\ k \in N$ *and let the series*

$$\sum_{k=1}^{\infty} k^{r-1}\ E_{k-1}(f)_{p(.)}$$

converges. Then f *is equivalent (equal almost everywhere) to a* $2\pi-$ *periodic absolutely continuous function* $\psi \in AC\,(\mathbb{T})$ *and the inequality*

$$\begin{aligned}&\left\|\psi^{(r)} - Z_{n,k}\left(\psi^{(r)}\right)\right\|_{L^{p(.)}(\mathbb{T})}\\ \leq\ & c_6(k,r)\left(\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_{\nu-1}(f)_{p(.)} + n^{-k}\sum_{\nu=1}^{n}\nu^{k+r-1}E_{\nu-1}(f)_{p(.)}\right),\ n \in \mathbb{N}\end{aligned}$$

holds.

Theorem 1.2. *Let* $f \in L^{p(.)}(\mathbb{T}),\ k \in N$. *Then the estimate*

$$\Omega_l\left(f, \frac{1}{n}\right)_{p(.)} \leq c_7\left\|f - Z_{n,k}(f)\right\|_{L^{p(.)}(\mathbb{T})} \tag{8}$$

holds, where $l = \{k, k-\text{oven},\ k+1, k-\text{odd}\}$.

Theorem 1.3. *Let* Γ *be Dini -smooth curve and* $p(.) \in \Phi_0^{\log}(\Gamma)$. *Then for* $f \in E^{p(.)}(G)$ *the following estimate holds:*

$$\begin{aligned}&\left\|f - Z_{n,k}(\cdot, f)\right\|_{L^{p(.)}(\Gamma)}\\ \leq\ & \frac{c_8(p)}{n^r}\left\{\sum_{\nu=0}^{n}\nu^{\gamma r-1}E_{\nu-1}(f)_{G,p(.)}\right\}^{1/\gamma}\end{aligned}$$

where $\gamma = \min\{2, p_*\}$.

The proof of the main results we need the following results.

Let $f \in E^{p(.)}(\mathbb{D})$. Applying Corollary 1 in the work [23] for the boundary values of $f \in E^{p(.)}(\mathbb{D})$ we have:

Lemma 1.1. *Let* $p(.) \in \Phi_0^{\log}(\mathbb{T})$ *and* $f \in E^{p(.)}(\mathbb{D})$. *Then the estimate*

$$\begin{aligned}&\left\|f - Z_{n,k}(\cdot, f)\right\|_{L^{p(.)}(\mathbb{T})}\\ \leq\ & \frac{c_9(p)}{n^r}\left\{\sum_{\nu=0}^{n}\nu^{\gamma r-1}E_{\nu-1}(f)_{\mathbb{D},\ p(.)}\right\}^{1/\gamma} n = 1, 2, \ldots,\end{aligned}$$

holds, where $\gamma = \min\{2, p_*\}$.

Lemma 1.2. *Let* Γ *be a Dini- smooth curve and* $p(.) \in \Phi_0^{\log}(\Gamma)$. *If* $f \in E^{p(.)}(G)$, *then*

$$E_n\left(f_0^+\right)_{\mathbb{D},\ p_0(.)} \leq c_{10}E_n(f)_{G,p(.)} \leq c_{11}E_n(f_0^+)_{\mathbb{D},p_0(.)}$$

with some positive constants c_{10} *and* c_{11} *independent of* n.

2. Proofs of Theorems

Proof of Theorem 1.1. According to [37, Theorem 6] f is equivalent (equal almost everywhere) to a $2\pi-$periodic absolutely continuous function $\psi \in AC(\mathbb{T})$ and the following inequality holds:

$$E_n(\psi^{(r)})_{p(.)} \leq c_{12}\left((n+1)^r E_n(f)_{p(.)} + \sum_{k=n+1} k^{r-1}E_k(f)_{p(.)}\right). \tag{9}$$

On the other hand the inequality

$$\|g - Z_{n,k}(g)\|_{L^{p(.)}(\mathbb{T})} \leq c_{13}n^{-k}\sum_{\nu=1}^{n} \nu^{k-1}E_{\nu-1}(g)_{p(.)} \tag{10}$$

holds [23]. Using (9) and (10) we get

$$\begin{aligned}
& \left\|\psi^{(r)} - Z_{n,k}(\psi^{(r)})\right\|_{L^{p(.)}(\mathbb{T})} \\
\leq\ & c_{14}n^{-k}\sum_{\nu=1}^{n} \nu^{k-1}E_{\nu-1}(\psi^{(r)})_{p(.)}, \\
\leq\ & c_{15}n^{-k}\sum_{\nu=1}^{n} \nu^{k+r-1}E_{\nu-1}(f)_{p(.)} \\
& +c_{16}n^{-k}\sum_{\nu=1}^{n} \nu^{k-1}\sum_{\mu=\nu+1}^{n} \mu^{r-1}E_{\mu-1}(f)_{p(.)} \\
& +c_{17}n^{-k}\sum_{\nu=1}^{n} \nu^{k-1}\sum_{\mu=n+1}^{\infty} \mu^{r-1}E_{\mu-1}(f)_{p(.)} \\
\leq\ & c_{18}n^{-k}\sum_{\nu=1}^{n} \nu^{k+r-1}E_{\nu-1}(f)_{p(.)} + c_{19}n^{-k}\sum_{\mu=1}^{n} \mu^{r-1}E_{\mu-1}(f)_{p(.)}\sum_{\nu=1}^{\mu} \nu^{k-1} \\
& +c_{20}\sum_{\mu=n+1}^{\infty} \mu^{r-1}E_{\mu-1}(f)_{p(.)} \\
\leq\ & c_{21}\left(\sum_{\nu=n+1}^{\infty} \nu^{r-1}E_{\nu-1}(f)_{p(.)} + n^{-k}\sum_{\nu=1}^{n} \nu^{k+r-1}E_{\nu-1}(f)_{p(.)}\right).
\end{aligned}$$

which completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let $T_n(f,x)$ be a trigonometric polynomial of best approximation to f in $L^{p(.)}(\mathbb{T})$. It is known that the following identity holds:

$$\begin{aligned}
& T_n(f,x) - Z_{n,k}(T_n(f),x) \\
=\ & f(x) - Z_{n,k}(f,x) + T_n(f,x) - f(x)
\end{aligned}$$

$$+Z_{n,k}(f-T_n(f),x). \tag{11}$$

Considering [33] we obtain

$$\|Z_{n,k}(f,\cdot)\|_{L^{p(\cdot)}(\mathbf{T})} \leq c_{22}\|f\|_{L^{p(\cdot)}(\mathbb{T})}. \tag{12}$$

Consideration of (11) and (12) gives us

$$\begin{aligned}
& \|T_n(f,x)-Z_{n,k}(T_n(f),x)\|_{L^{p(\cdot)}(\mathbb{T})} \\
\leq\ & \|f-Z_{n,k}(f)\|_{L^{p(\cdot)}(\mathbb{T})}+\|T_n(f)-f\|_{L^{p(\cdot)}(\mathbb{T})} \\
& +\|Z_{n,k}(f-T_n(f))\|_{L^{p(\cdot)}(\mathbb{T})} \\
\leq\ & \|f-Z_{n,k}(f)\|_{L^{p(\cdot)}(\mathbb{T})}+c_{23}E_n(f)_{p(.)} \\
\leq\ & c_{24}\|f-Z_{n,k}(f)\|_{L^{p(\cdot)}(\mathbb{T})}. \qquad (13)
\end{aligned}$$

If $k-$is an even number the following relation holds:

$$T_n(f,x)-Z_{n,k}(T_n(f),x)=(-1)^{\frac{k}{2}}(n+1)^{-k}T_n^{(k)}(f,x). \tag{14}$$

Then using (13),(14) and [37, Corollary 2] we get

$$\begin{aligned}
& \Omega_k\left(f,\frac{1}{n}\right)_{p(.)} \\
=\ & \Omega_k\left(f-T_n(f)+T_n(f),\frac{1}{n}\right)_{p(.)} \\
\leq\ & c_{25}\Omega_k\left(f-T_n(f)\right)_{p(.)}+c_{26}\Omega_k\left(T_n(f),\frac{1}{n}\right)_{p(.)} \\
\leq\ & c_{27}\|f-T_n(f)\|_{p(.)}+c_{28}n^{-k}\left\|T_n^{(k)}(f)\right\|_{L^{p(\cdot)}(\mathbb{T})} \\
\leq\ & c_{29}E_n(f)_{p(.)}+c_{30}\|T_n(f)-Z_{n,k}(T_n(f))\|_{L^{p(\cdot)}(\mathbb{T})} \\
\leq\ & c_{31}E_n(f)_{p(.)}+c_{32}\|f-Z_{n,k}(f)\|_{L^{p(\cdot)}(\mathbb{T})} \\
\leq\ & c_{33}\|f-Z_{n,k}(f)\|_{L^{p(\cdot)}(\mathbb{T})}. \qquad (15)
\end{aligned}$$

Let $\widetilde{T_n}^{(k)}(f,x)$ be a trigonometric conjugate of $T^{(k+1)}(f,x)$. If k is a odd number the relation

$$T_n(f,x)-Z_{n,k}(T_n(f),x)=(-1)^{\frac{k+3}{2}}(n+1)^{-k}\widetilde{T_n}^{(k)}(f,x). \tag{16}$$

holds. Also, according to [37] we obtain

$$\left\|T^{(k+1)}(f,x)\right\|_{L^{p(\cdot)}(\mathbb{T})} \leq c_{34}n\left\|\widetilde{T_n}^{(k)}(f)\right\|_{L^{p(\cdot)}(\mathbb{T})}. \tag{17}$$

Use of (16), (17) gives us

$$\Omega_{k+1}\left(f,\frac{1}{n}\right)_{p(.)}$$

$$
\begin{aligned}
& = \Omega_{k+1}\left(f-T_n(f)+T_n(f),\frac{1}{n}\right)_{p(.)} \\
& \leq c_{35}\Omega_{k+1}\left(f-T_n(f)\right)_{p(.)}+c_{36}\Omega_{k+1}\left(T_n(f),\frac{1}{n}\right)_{p(.)} \\
& \leq c_{37}\left\|f-T_n(f)\right\|_{p(.)}+c_{38}n^{-(k+1)}\left\|T_n^{(k+1)}(f)\right\|_{L^{p(\cdot)}(\mathbb{T})} \\
& \leq c_{39}E_n(f)_{p(.)}+c_{40}n^{-k}\left\|\widetilde{T_n}^{(k)}(f)\right\|_{L^{p(\cdot)}(\mathbb{T})} \\
& \leq c_{41}E_n(f)_{p(.)}+c_{42}\left\|T_n(f)-Z_{n,k}\left(T_n(f)\right)\right\|_{L^{p(\cdot)}(\mathbb{T})} \\
& \leq c_{41}E_n(f)_{p(.)}+c_{43}\left\|f-Z_{n,k}\left(f\right)\right\|_{L^{p(\cdot)}(\mathbb{T})} \\
& \leq c_{44}\left\|f-Z_{n,k}\left(f\right)\right\|_{L^{p(\cdot)}(\mathbb{T})}. \qquad (18)
\end{aligned}
$$

From (15) and (18) we obtain inequality (8). Thus, the proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. Let $f\in E^{p(\cdot)}(\mathbb{G})$. The function f has the Faber series

$$f(z)\sim\sum_{k=0}^{\infty}c_k(f)\varphi_k(z)$$

Then by [18, Lemma 1] $f_0^{+}\in E^{p_0(\cdot)}(\mathbb{D})$ and for the function f_0^{+} the Taylor expansion

$$f_0^{+}(t)\sim\sum_{k=0}^{\infty}c_k(f)w^k, w\in\mathbb{U}$$

holds. According to [5, p. 38, Theorem 3.4] boundary function $f_0^{+}\in L^{p_0(\cdot)}(\mathbb{T})$ has the Fourier expansion

$$f_0^{+}(t)\sim\sum_{k=0}^{\infty}c_k(f)e^{ikt}.$$

Using the boundedness of the operators $T:E^{p_0(\cdot)}(\mathbb{D})\to E^{p(\cdot)}(\mathbb{G})$, $T^{-1}:E^{p(\cdot)}(\mathbb{G})\to E^{p_0(\cdot)}(\mathbb{D})$ [18], Lemma 1.1 and 1.2 we obtain

$$
\begin{aligned}
& \left\|f-Z_{n,k}\left(\cdot,f\right)\right\|_{L^{p(\cdot)}(\lneqq)} \\
\leq & \left\|T\left(f_0^{+}\right)-T\left(Z_{n,k}\left(\cdot,f_0^{+}\right)\right)\right\|_{L^{p(\cdot)}(\Gamma)} \\
\leq & \left\|f_0^{+}-Z_{n,k}\left(\cdot,f_0^{+}\right)\right\|_{L^{p_0(\cdot)}(\mathbb{T})} \\
\leq & \frac{c_{45}\left(p\right)}{n^r}\left\{\sum_{\nu=0}^{n}\nu^{\gamma r-1}E_{\nu-1}\left(f_0^{+}\right)_{\mathbb{D},\ p(.)}\right\}^{1/\gamma} \\
\leq & \frac{c_{46}\left(p\right)}{n^r}\left\|T^{-1}\right\|\left\{\sum_{\nu=0}^{n}\nu^{\gamma r-1}E_{\nu-1}\left(f_0^{+}\right)_{\mathbb{D},\ p_0(.)}\right\}^{1/\gamma}
\end{aligned}
$$

$$\leq \frac{c_{47}(p)}{n^r}\left\{\sum_{\nu=0}^{n}\nu^{\gamma r-1}E_{\nu-1}\left(f_0^+\right)_{\mathbb{G},\ p(.)}\right\}^{1/\gamma}$$

which completes the proof of theorem 1.3. □

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